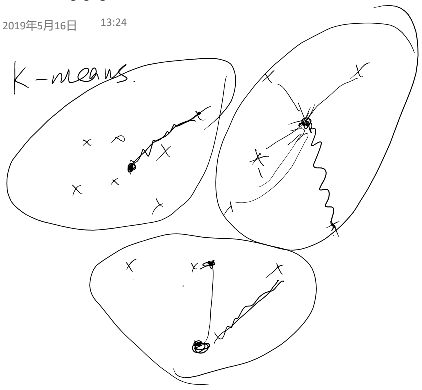
2019 Lec 9

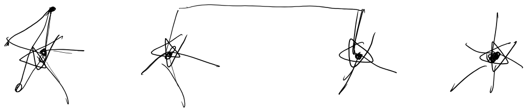
2019年5月16日 13:24

K-means.



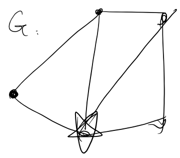
k- center.

OPT



Dominating set.

G.

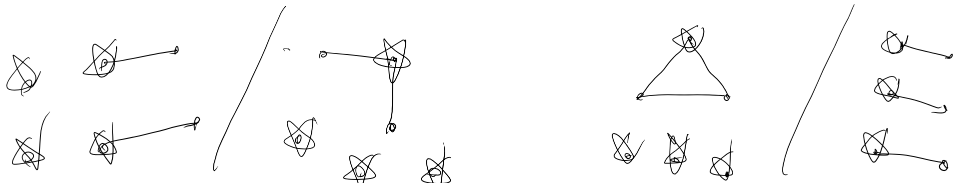


Dom(G) = 1

$w(e_1) \le w(e_2) \le w(e_3) \cdots \le w(e_n)$

	$E=\phi$	$E=\{e_1\}$	$E=\{e_1, e_2\}$		e_{i-1}	$E=\{e_1,\cdots,e_i\}$		$E=\{e_1,\cdots,e_m\}$
	G_0	G_1	G_2	G_3	G_{i-1}	G_i	- - -	$G_m = G$
min #DomS	n	n-1	n-2	n-2/n-3	$>k$	$\le k$		

$\longrightarrow$ OPT $= w(e_i)$



OPT $= w(e_i)$ $w(e_j)$

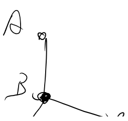
	G_0	G_1	G_2	G_{l-1}	G_l	G_{i-1}	G_i	G_{j-1}	G_j
min #DomSet	n	n-1	n-2			$\boxed{>k}$	$\le k$		$\ge k$
Approx Dom						$>k$	$>k$ / $\le k$	$>k$	$\le k$
	G_0^2	$G_1^2 = G_1$	G_2^2			G_{i-1}^2	G_i^2		
max Independent set	n	n-1				$>k$ or $\le k$	$\le k$		
maximal Ind Set	n	n-1		$>k$	ALGO $\boxed{\le k}$		$\le k$		

$\forall H$ $\boxed{\text{max Ind Set}(H^2) \le \text{min Domset}(H)}$

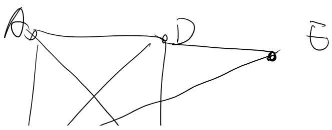
if $(u,v), (v,w) \in E(H) \Rightarrow (u,w) \in E(H^2)$

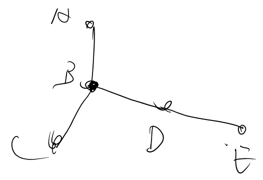
$(u,v) \in E(H) \Rightarrow (u,v) \in E(H^2)$

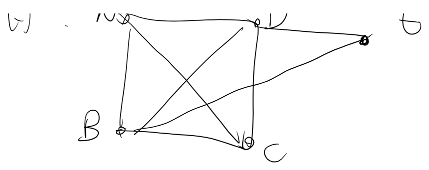
H

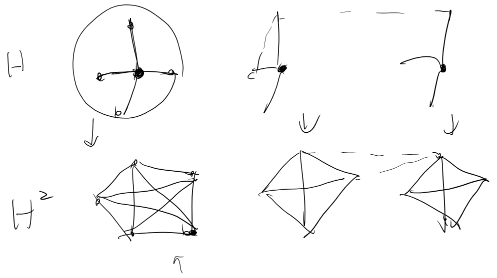


H^2









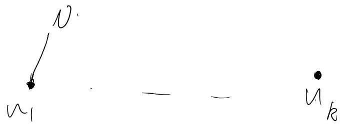
$\underset{\sim\sim}{w(e_l)}$

G_0 . — G_{l-1} G_l

Domset

G_0^2 . — G_{l-1}^2 G_l^2

maximal Ind Set . $>k$ $\boxed{\leq k}$

G_l^2 

G_l 

(u,v) 可能不在 G_l

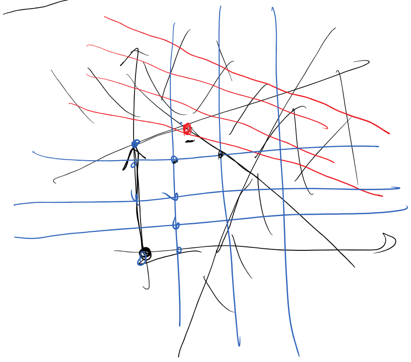
$w(u_1,v)$

$\leq w(u_1,x)+w(x,v)$

$\leq 2\,w(e_l)$

Algo $\leq 2\,w(e_l) \leq 2\cdot OPT$

$w(e_l) \leq w(e_i) = OPT$



x, y

$ax+by \geq c$

Vertex cover (weighted)

LP

$$\min \sum w(v)\, x_v$$

$$\text{s.t.} \quad x_i + x_j \geq 1 \quad \forall (i,j) \in E$$

s.t. $x_i + x_j \geq 1 \quad \forall (i,j) \in E$

~~$x_i \in \{0,1\}$~~ $\quad 0 \leq x_i \boxed{\leq 1}$

LP. OPT. $(0.3,\ 0.7,\ 0.8,\ 0.6 \ldots)$

Algo: $(0\quad 1\quad 1\quad 1 \quad)$

$w(\text{Algo}) \leq 2\, OPT(LP) \leq 2 \cdot OPT(IP)$

$OPT(LP) \leq OPT(IP)$

n-clique. IP. $\sum x_v$

s.t. $x_i + x_j \geq 1 \quad \forall i \neq j$.

$x_i \in \{0,1\}$

$OPT(IP) = n-1$

Gap ≈ 2

LP $\quad \sum x_v$

s.t. $x_i + x_j \geq 1 \quad \forall i \neq j$

$0 \leq x_i \leq 1$

$OPT(LP) = \frac{n}{2}$

$x_i = \frac{1}{2}$

$\sum x_v$

s.t. $2x_i + 2x_j \geq 1 \quad (i,j) \in E$

$0 \leq x_i \leq 1$

$OPT'(LP) = \frac{n}{4}$

Gap ≈ 4

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weak duality.

$\sum c_j x_j \geq OPT(\text{prime}) = OPT(\text{dual}) \geq \sum b_i y_i$

$\text{Algo} \leq \alpha \cdot OPT(LP)$

$\boxed{OPT(LP)} \geq \frac{\text{Algo}}{\alpha}$

prime

s.t. $\sum_j a_{ij} x_j \geq b_i \longleftrightarrow y_i \geq 0$

$y_i \cdot (\sum_j a_{ij} x_j - b_i) = 0$

$\sum_i a_{ij} y_i \leq c_j \longleftrightarrow x_j \geq 0$

Set Cover

Primal LP

$\min \sum_{s \in \mathcal{S}} c(S) X_s$

s.t. $\sum_{S: e \in S} X_S \geq 1 \ (\forall e \in U)$

$X_S \geq 0$

Dual

$\max \sum_{e \in U} y_e$

s.t. $\sum_{e: e \in S} y_e \leq C(S) \ (\forall S \in \mathcal{S})$

$y_e \geq 0$

$\max \dfrac{C(S)}{|S - \text{已经 covered 元素}|}$

Algo: Greedy algorithm.

1° $S_1 = \arg\min \dfrac{c(S)}{|S|}$

$\forall e \in S_1 \quad y_e = \dfrac{c(S_1)}{|S_1|}$

2° $S_2 = \arg\min \dfrac{c(S)}{|S - S_1|}$

$\forall e \in S_2 - S_1 \quad y_e = \dfrac{c(S_2)}{|S_2 - S_1|}$

⋮

$\sum y_e$

$= \text{Algo}$

$= c(S_1) + c(S_2) + \cdots$

Thm: $y'_e = \dfrac{y_e}{\log n}$. $\{y'_e\}$ is a feasible solution of Dual.

$$\sum y_e' \le OPT(LP) \le OPT(IP)$$

$$\frac{Algo}{\log n} = \frac{\sum y_e}{\log n} \le OPT(IP)$$

proof: $\forall S.$ $\sum_{e \in S} y_e \le c(S) \log n.$

$S = \{e_1, e_2, \cdots e_k\}$

greedy algo: S' cover e_1. $\quad y_{e_1} = \frac{c(S')}{|S' - \text{已经cover element}|}$

$$\le \frac{c(S)}{|S|} = \frac{c(S)}{k}.$$

S'' cover e_i $\quad y_{e_i} = \frac{c(S'')}{|S'' - \text{已经 covered}|}$

$$\le \frac{c(S)}{|S - \text{已经 covered}|}$$

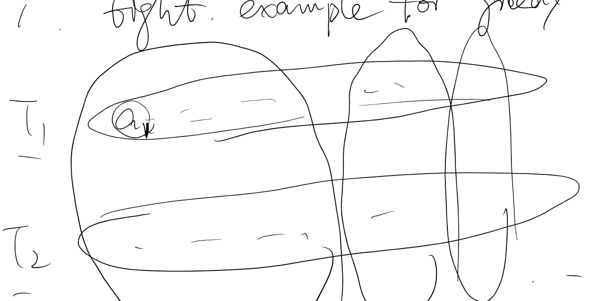
$$\le \frac{c(S)}{k-(i-1)}.$$

$$\sum_{i=1}^{k} y_{e_i} \le c(S) \cdot \left(\frac{1}{k} + \frac{1}{k-1} + \cdots + 1\right)$$

$$\le c(S) \cdot \log k \le c(S) \cdot \log n.$$

Integrality Gap

1° tight example for greedy algo.



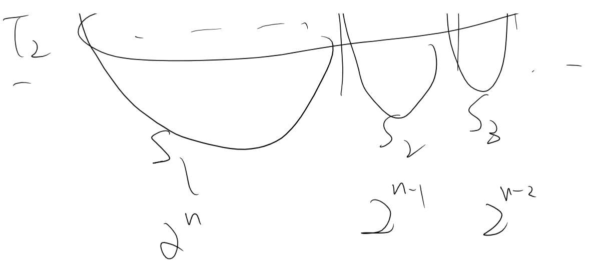
$2^n - 1$

$\min \sum X_S$

s.t. $X_{T_1} + X_{S_1} \ge 1 \quad X_{T_2} + X_{S_1} \ge 1$

$X_{T_1} + X_{S_2} \ge 1 \quad X_{T_2} + X_{S_2} \ge 1$

$\vdots$



$x_{T_1} + x_{S_n} \ge 1$ $\quad$ $x_{T_2} + x_{S_n} \ge 1$

IP OPT $= 2$ ✓

LP OPT $= 2$

2° Integrality gap $= \log n$

→ ~~element~~ : $\{e_1, e_2, \cdots, e_n\}$ $\quad n = 2^k - 1$

→ set: $S_i = \{ e_j \mid i \cdot j = 1 \}$ (over F_2) $\quad i = 1, 2, \cdots, n$

$S_1 = S_{\underbrace{0\cdots0}_{k-1}1} = \{ e_{\underbrace{xx\cdots x}_{k-1}1} \}$ $\quad |S_1| = 2^{k-1}$

$\forall i \quad |S_i| = 2^{k-1}$

$\forall e \quad \#\{ S_i \mid e \in S_i \} = 2^{k-1}$

min $\sum x_{S_i}$

s.t. $\forall e \quad \underbrace{x_{S_i} + \cdots + x_{S_j}}_{2^{k-1}\text{ 项}} \ge 1$

$x_{S_i} = \frac{1}{2^{k-1}}$

$OPT(LP) \le \frac{n}{2^{k-1}} \approx 2$

Thm $OPT(IP) \ge k$

$S_{i_1}, S_{i_2}, \cdots S_{i_p}$ $\quad \exists j$

$\Leftrightarrow \quad i_1 \cdot j = 0, \quad i_2 \cdot j = 0 \quad \cdots \quad i_p \cdot j = 0$

$A \cdot \vec{j} = 0 \qquad A: p \times k$ matrix

$\text{rank}(A) < k$ $\quad$ $p < k$